Topic - Problem based on 2nd order Differential Equation with variable Coefficient - (Method of changing the independent variable) contd
B.Sc III (Math HONS)

Date -

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Solve $x^4 \frac{d^2y}{dx^2} + 2x^3 \frac{dy}{dx} + a^2 y = 0$

Here given equation is

$$x^4 \frac{d^2y}{dx^2} + 2x^3 \frac{dy}{dx} + a^2 y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{a^2}{x^4} y = 0.$$

Here $P = \frac{2}{x}$ $\quad Q = \frac{a^2}{4}$ $\quad R = 0$

Changing the independent variable from x to z, the equation becomes

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = 0$$

Where $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$ $\qquad Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2} = a^2$

$\cancel{Q_1 = a^2 = \dfrac{\frac{a^2}{4}}{\left(\frac{dz}{dx}\right)^2}}$

$\cancel{a^2 \left(\frac{dz}{dx}\right)^2 = a^2}$

$$a^2 \left(\frac{dz}{dx}\right)^2 = \frac{a^2}{x^4}$$

$$\left(\frac{dz}{dx}\right)^2 = \frac{1}{x^4}$$

$$\therefore \frac{dz}{dx} = \frac{1}{x^2}$$

$$\Rightarrow \int dz = \int x^{-2} dx$$

$$z = -\frac{1}{x}$$

$$\frac{dz}{dx} = \frac{1}{x^2} \qquad \frac{d^2 z}{dx^2} = -\frac{2}{x^3}$$

$$\text{then } P_1 = \frac{-\frac{2}{x^3} + \frac{2}{x} \cdot \frac{1}{x^2}}{\left(\frac{dz}{dx}\right)^2} = 0$$

thus Transformed equation is

$$\frac{d^2y}{dz^2} + a^2 y = 0$$

Auxiliary Equation

$$D^2 + a^2 = 0$$

$$D^2 = -a^2 = a^2 i^2$$

$$D = \pm a i$$

$$\therefore \text{Solution} = c_1 \cos az + c_2 \sin az$$

$$= c_1 \cos a\left(\frac{-1}{x}\right) + c_2 \sin a(-1/x)$$

$$= c_1 \cos a/x - c_2 \sin \frac{a}{x}$$

solve the Differential Equation

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin x^2$$

solution – Given Equation is

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin x^2$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2\sin x^2$$

Here $P = -1/x$ $Q = -4x^2$

$R = 8x^2\sin x^2$

changing the independent variable from x to z, the equation reduces to –

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + Q_1y = R_1$$

Where $P_1 = \dfrac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{(\frac{dz}{dx})^2}$ $Q_1 = \dfrac{Q}{(\frac{dz}{dx})^2}$

$$R_1 = \frac{R}{(\frac{dz}{dx})^2}$$

Now $\dfrac{Q}{(\frac{dz}{dx})^2} =$

$$\Rightarrow \frac{-\cancel{4}x^2}{(\frac{dz}{dx})^2} = -4 \Rightarrow x^2 = (\frac{dz}{dx})^2$$

$\therefore \dfrac{dz}{dx} = x$, $\dfrac{d^2z}{dx^2} = 1$

$dz = x\,dz$

$z = x^2/2$

Now $P_1 = \dfrac{P\dfrac{dz}{dx} + \dfrac{d^2z}{dx^2}}{\left(\dfrac{dz}{dx}\right)^2}$

$= \dfrac{-\frac{1}{x} \times x + 1}{\left(\dfrac{dz}{dx}\right)^2} = 0$

$Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2} = -\dfrac{4x^2}{x^2} = -4$

$R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2} = \dfrac{8x^2 \sin x^2}{x^2} = 8 \sin x$

$\therefore$ the Differential Equation reduces

$\dfrac{d^2y}{dz^2} + P_1\dfrac{dy}{dz} + Q_1 y = R_1$

$\dfrac{d^2y}{dz^2} + (-4)y = \dfrac{8 \sin x^2}{4} = 2 \sin z$

$\dfrac{d^2y}{dz^2} - 4y = 2 \sin z.$

Auxiliary Equation

$D^2 - 1 = 0$

$D^2 = 1 \quad D = \pm 1$

$\therefore y = c_1 e^{z} + c_2 e^{-z} - \sin z = c_1 e^{x^2} + c_2 e^{-x^2} \ldots$